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Damped vibration of a string

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The bounded solution of the unsteady Stokes equations is obtained for the flow of a viscous incompressible fluid about a circular cylinder which undergoes a linear translation starting from rest. A drag formula which consists of the known added-mass term and an additional term arising from the presence of viscosity is obtained. The drag obtained is applied locally in a study of damped vibration of a string. It is shown that the usual theory based on the quasi-steady drag formula overestimates considerably the period and the decay rate of damped vibration of a string in a viscous fluid.

1. Introduction

A good knowledge of unsteady Stokes flows is useful for understanding some important aspects of suspension rheology, micro-organism propulsion and hot-wire instrumentation.

The unsteady flow obtained by Basset (1888) for a sphere translating in a viscous fluid is well known. This solution of the unsteady Stokes equations is uniformly valid in time and reduces to the known steady solution of Stokes if the translational velocity of the sphere attains a constant value. Ockendon (1968) considered the solution of the Navier–Stokes equations for the flow produced by a sphere which starts from rest and eventually reaches a constant velocity. He expressed the solution as an asymptotic expansion in terms of the (small) Reynolds number and showed that the expansion becomes invalid for large times unless the time scale on which the sphere's velocity varies is sufficiently large.

Batchelor (1954) and Hasimoto (1956) obtained a bounded solution uniformly valid in time of the unsteady Stokes equations in their studies of the flow that is generated by forced motion of an infinite cylinder parallel to its length in a viscous incompressible fluid. On the other hand it is well known that the solution of the Stokes equations for a circular cylinder steadily translating in a direction normal to its axis does not satisfy the boundary condition at infinity. Proudman & Pearson (1957) and Kaplun (1957) have shown that this Stokes solution can be matched to the small Reynolds number asymptotic solution of the Navier–Stokes equations that is valid in a region far away from the cylinder where the convective acceleration is as important as the viscous diffusion. However the transient flow which approaches the above steady solution has not yet been given.

It is known that the unsteady Stokes equations describe high frequency, small amplitude oscillatory flows at all Reynolds numbers as well as the initial development of the flow of a viscous fluid accelerated rapidly from rest. (See Batchelor 1967, pp. 216, 353; Landau & Lifshitz 1959; Stuart 1963.) Despite their wide application, only a few solutions of the unsteady Stokes equations are known for flows which start from rest. A bounded solution of the unsteady Stokes equations is given in the next section for the flow around a circular cylinder which undergoes a linear translatory oscillation. The oscillation is assumed to be along a straight line normal to the axis of the cylinder but is otherwise quite general. The corresponding transient drag on the cylinder is also given. The drag formula consists of two terms of equal magnitude. The first is the familiar added-mass term while the second arises from the action of viscosity. The results are applied in §3 to calculate the transient viscous damping of a string which is displaced to a given initial position before being released from rest.

2. Unsteady Stokes flow

Consider the flow of a viscous incompressible fluid around a body. The motion of the body is described by the velocity vector $\mathbf{u}(t) = -\mathbf{i}u(t)$, where t denotes time and $\mathbf{i}$ is a unit vector. The governing equations of the flow relative to a reference frame attached to the cylinder are $\nabla.\mathbf{V} = 0$ and

$$\partial\mathbf{V}/\partial t + (\mathbf{V}.\nabla)\mathbf{V} = -\rho^{-1}\nabla p + \nu(\nabla.\nabla)\mathbf{V} + \mathbf{i}du/dt, \tag{1}$$

where $\mathbf{V}$ is the velocity, p the dynamic pressure, ρ the density, ν the kinematic viscosity and ∇ is the gradient operator (see Batchelor 1967, p. 140). The above nonlinear equation can be linearized near the body in two important limiting cases of oscillatory motion in a viscous fluid (cf. Landau & Lifshitz 1959, p. 88; Batchelor 1967, pp. 216, 353). The first limiting case is that of a low frequency oscillation such that $na^2/\nu \ll 1$ and $na\delta/\nu \ll 1$, where n, a and δ are respectively the characteristic frequency, a representative dimension of the body and the amplitude of the oscillation. For this case the flow is quasi-steady and is governed by the steady Stokes equations. The second limiting case is that of a small amplitude oscillation such that $\delta \ll a$ but na^2/ν is not necessarily smaller than one. For this case the flow is governed by the unsteady Stokes equations even if the Reynolds number $na\delta/\nu$ is greater than one. Note that the subcase of the second case when $na^2/\nu \ll 1$ is also covered by the first limiting case. The unsteady Stokes equations also describe the initial development of flows due to rapid acceleration of a body initially at rest (see Stuart 1963; Lin 1976) if $\delta \ll a$. For these flows the governing equations are

$$(\partial/\partial\tau - \nabla_1.\nabla_1)\nabla_1 \times \mathbf{V}_1 = 0, \quad \nabla_1.\mathbf{V}_1 = 0, \tag{2a,b}$$

where the time, the gradient operator and the velocity have been respectively normalized according to $\tau(a^2/\nu) = t$, $\nabla_1 = a\nabla$ and $\mathbf{V}_1 U = \mathbf{V}$, U being the maximum velocity of the cylinder. The corresponding boundary conditions are that the fluid should stick to the surface of the body and that the velocity $\mathbf{V}_1$ should tend to $\mathbf{i}u/U$ at infinity. The fluid is supposed to be initially motionless.

Consider the special case of two-dimensional flow around a circular cylinder of radius a. For this case, it is convenient to use the polar co-ordinates (r, θ), where r is the radial distance measured from the axis of the cylinder and θ is the angle measured counterclockwise from $\mathbf{i}$. Equation (2*b*) enables one to define a stream function ψ in terms of which the r and θ components of the velocity vector can be written respectively as

$$V_{1r} = -r^{-1}\,\partial\psi/\partial\theta, \quad V_{1\theta} = \partial\psi/\partial r.$$

The vorticity has only one component, in the z direction, normal to the r, θ plane, and is given by

$$\zeta_1 = \nabla_1 \times \mathbf{V}_1 = \mathbf{i}_z \nabla^2\psi,$$

where ∇^2 is the Laplacian defined by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\partial\theta^2},$$

the $\mathbf{i}$'s are unit vectors and the subscripts denote directions. Substitution of the above expressions for the vorticity and the Laplacian operator into (2*a*) yields the diffusion equation for the vorticity:

$$(\partial/\partial\tau - \nabla^2)\,\nabla^2\psi = 0. \tag{3}$$

The initial condition for this equation is

$$\psi(r, \theta, 0) = 0.$$

The boundary conditions are

$$\partial\psi/\partial\theta = \partial\psi/\partial r = 0 \quad \text{at} \quad r = 1,$$

$$-r^{-1}\partial\psi/\partial\theta = [v(\tau) + f(r,\tau)]\cos\theta, \quad \partial\psi/\partial r = [-v(\tau) + g(r,\tau)]\sin\theta \quad \text{as} \quad r \to \infty,$$

where $v(\tau) = u/U$, and $f(r,\tau)$ and $g(r,\tau)$ are as yet unknown functions which must remain smaller than order one if the boundary condition at infinity is to be satisfied to first order. The solution of (3) with the above conditions has been obtained already by Lin (1976) in his study of the initial drag on a cylinder and will be only briefly mentioned here.

The solution of (3) that satisfies the boundary condition $\partial\psi/\partial\theta = 0$ at $r = 1$ can be written as

$$\psi = \left[v(\tau)\left(\frac{1}{r} - r\right) + \frac{1}{r}\int_1^r \chi(s,\tau)\,s\,ds\right]\sin\theta = \psi_0 \sin\theta. \tag{4}$$

The first term in the right-hand side of this equation corresponds to the irrotational part of the solution and the term involving the integral arises from the action of viscosity. Transformations similar to (4) have been used by Lin & Gautesen (1972) for the creeping flow around a deforming sphere. Substitution of the above form of the solution into (3) and its initial and boundary conditions shows that $\chi(r,\tau)$ must satisfy the diffusion equation

$$\frac{\partial\chi}{\partial\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\chi}{\partial r}\right)$$

subject to the conditions

$$\chi(r,0) = 0, \quad \chi(1,\tau) = 2v(\tau), \quad \lim_{r\to\infty}\chi(r,\tau) = o(1).$$

The solution for $\chi(r,\tau)$ may easily be obtained by the method of Laplace transforms and is given by

$$\chi(r,\tau) = 2\int_0^\tau \dot{v}(\lambda)\,\bar{\chi}[r,(\tau-\lambda)]\,d\lambda, \tag{5}$$

where $$\bar{\chi}(r,\tau) = 1+\frac{2}{\pi}\int_0^\infty \exp(-s^2\tau)\frac{J_0(sr)\,Y_0(s)-Y_0(sr)\,J_0(s)}{J_0^2(s)+Y_0^2(s)}\frac{ds}{s}$$

and the dot denotes time differentiation. This function also describes the transient temperature field caused by heat conduction when a circular cylinder in an unbounded medium with zero initial temperature is subjected to a unit step temperature increase (cf. Carslaw & Jaeger 1959, pp. 335–336). In the above expression J_0 and Y_0 stand for the zeroth-order Bessel functions of the first and second kinds respectively.

By using (4), the functions f and g can be expressed in terms of χ as

$$f = \lim_{r\to\infty} -\frac{1}{r^2}\left[\int_1^r \chi(s,\tau)\,s\,ds + v(\tau)\right],$$

$$g = f + \lim_{r\to\infty}\chi(r,\tau).$$

It was mentioned earlier that, as long as f and g remain $o(1)$, the solution (4) can be regarded as uniformly valid in space. We expect the condition $f, g = o(1)$ to be satisfied in unsteady oscillatory Stokes flows. The general results (4) and (5) are applied to the problem of forced transient oscillation of a cylinder in the appendix. In this particular example both g and f vanish as $r\to\infty$.

The drag force $\mathbf{D}$ per unit length for a cylinder translating at the velocity $-\mathbf{i}v(\tau)$ can be calculated from

$$\mathbf{D} = \mathbf{i}\int_0^{2\pi} a(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\,d\theta,$$

where the integration is along $r = 1$ and where σ_{rr} and $\sigma_{r\theta}$ are the normal and tangential stress components, related to the velocity field by

$$\sigma_{rr} = \frac{\rho\nu U}{a}\left(-p_1 + \frac{2}{r^2}\psi_\theta - \frac{2}{r}\psi_{r\theta}\right),$$

$$\sigma_{r\theta} = \frac{\rho\nu U}{a}\left[r\left(\frac{\psi_r}{r}\right)_r - \frac{1}{r^2}\psi_{\theta\theta}\right],$$

where p_1 is the pressure normalized by $\rho\nu U/a$ and the subscripts on the right-hand sides of the above equations denote partial differentiations. From the θ component of the Stokes equations, one has

$$p_1 = [r\psi_{0r\tau} - 2r^{-2}\psi_0 + 2r^{-1}\psi_{0r} - \psi_{0rr} - r\psi_{0rrr} + r\dot{v}]\cos\theta.$$

Thus, in terms of the stream function the drag is given by

$$\begin{aligned}\frac{\mathbf{D}}{\rho\nu U} &= \mathbf{i}\int_0^{2\pi}[(\psi_{0rrr}+2\psi_{0rr}-5\psi_{0r}-\psi_{0r\tau}-\dot{v})\cos^2\theta+(\psi_{0r}-\psi_{0rr})]\,d\theta \\ &= \mathbf{i}\pi(\psi_{0rrr}-\psi_{0r\tau}-3\psi_{0r}-\dot{v}) \\ &= \mathbf{i}\pi[\dot{v}-2\chi_r(1,\tau)],\end{aligned} \tag{6}$$

where $$\chi_r(1,\tau) = -\frac{8}{\pi^2}\int_0^\tau\left\{\int_0^\infty \dot{v}(\lambda)\frac{\exp[-s^2(\tau-\lambda)]}{s[J_0^2(s)+Y_0^2(s)]}ds\right\}d\lambda.$$

The first term on the right-hand side of (6), i.e. $\pi\dot{v}$, is the well-known added-mass term (cf. Milne-Thomson 1960, p. 237) and the second term is the additional drag due to viscosity. It is clear from the expression for $\chi_r(1,\tau)$ in (6) that if $\dot{v}$ does not change sign after the start then $\chi_r(1,\tau)$ and $\dot{v}$ are always of opposite sign. Hence, according to (6), the drag on a cylinder which accelerates or decelerates monotonically from rest in a viscous fluid is always larger by an amount $|2\pi\chi_r(1,\tau)|$ than that which the same cylinder would experience in an inviscid fluid. However, it must be pointed out that, if $\dot{v}$ is monotonic, then the use of the unsteady Stokes equations is justified only during the initial period when the distance travelled by the cylinder is much smaller than its diameter (see Lin 1976).

3. Damped vibration of a string

The forces on a cylinder and also on a sphere executing an undamped oscillation were obtained by Stokes (1851). The correction to the force on a sphere due to the damping was given by Meyer (1871) and Hussey & Vujacic (1967). The latter authors also gave the correction for a cylinder. However, these corrections were based on a prescribed exponential decay of the amplitude.

In practice, the fluid motion is set up from rest, and for some time after the initiation of the motion, the velocity field contains 'transients' determined by the initial conditions as is demonstrated in the last section. After this initial stage of flow development, it is usually assumed that the flow becomes quasi-steady and the damping force on the body becomes in phase with its oscillatory velocity. Then the rate of the decay of the oscillation due to viscosity can be obtained from consideration of the energy dissipation in the locally two-dimensional quasi-steady boundary layer over the body surface, if the Reynolds number of the flow is so large that boundary-layer theory is applicable (Batchelor 1967, pp. 355–358). Here we are interested in small amplitude oscillation including the initial stage of unsteady viscous damping for which the boundary-layer approximation may not be applicable. The numerical example given below shows that the quasi-steady theory overestimates considerably both the period and the decay rate of damped vibration of a string.

Consider a string of radius a stretched along the x axis between $x = 0$ and $x = la$ under a tension of magnitude S. Small amplitude vibration of such a string is governed by (cf. Rayleigh 1945, p. 177)

$$\pi a^2\rho_s\, y_{tt}(x,t) = S y_{xx}(x,t) + D(x,t), \tag{7}$$

where ρ_s is the (volume) density of the string, $y(x,t)$ the transverse displacement of the string at the station x at time t, D the drag per unit length of the string and the subscripts denote differentiations. The drag force given in the last section for an infinitely long circular cylinder is applied locally to the present problem of viscous damping of a string. Thus we put

$$D = \pi\rho\nu U[v(x,\tau)_\tau - 2\chi_\tau(1,x,\tau)].$$

This approximation is expected to be valid if the amplitude is much smaller than the wavelength for all Fourier components required to describe the vibration of the string.

Using the dimensionless variables $x = laX$, $y = aZ$ and $t = T/\Omega$, where Ω is a characteristic frequency of the vibration, we write (7) as

$$Z_{XX} - Z_{TT} = \epsilon \int_0^T \int_0^\infty \left[\frac{\partial^2 Z}{\partial T^2}\right]_{T=\lambda} \frac{\exp[-s^2(T-\lambda)]}{[J_0^2(s)+Y_0^2(s)]\,s}\, ds\, d\lambda, \tag{8}$$

in which $$\epsilon = \frac{16}{\pi^2}\left(\frac{\rho}{\rho+\rho_s}\right)\left(\frac{\nu}{\Omega a^2}\right), \quad \Omega = \left[\frac{S}{\pi(\rho+\rho_s)}\right]^{\frac{1}{2}}\left(\frac{1}{la^2}\right).$$

Suppose that the string is fixed at both ends and is displaced to a given initial position before being released from rest at $T = 0$. The initial condition for each Fourier component of the vibration is then

$$Z(X,0) = h_N \sin(N\pi X), \quad Z_T(X,0) = 0, \tag{9}$$

where $h_N a$ is the initial amplitude and N is a positive integer.

Recall that the drag force employed to arrive at (8) was obtained through the linearization of the Navier–Stokes equations for small amplitude oscillatory flows. Therefore (8) is applicable only if $h_N \ll 1$ but na^2/ν need not be smaller than one. If na^2/ν is also smaller than one, then the flow is quasi-steady and (7) can be reduced to equation (12) below. Moreover, for the local substitution of (6) in (7) to be reasonable, we must have $la/N \gg a$ for all N necessary to describe the motion adequately. Thus, for the vibration of a string the description of whose shape at any time requires more than the first l terms in the Fourier series, (8) is not applicable.

The solution of (8) under the above conditions can be written as

$$Z = h_N z_N(T) \sin(N\pi X),$$

where $z_N(T)$ must satisfy the integro-differential equation

$$\ddot{z}_N(T) + (N\pi)^2 z_N = -\epsilon \int_0^T \int_0^\infty \ddot{z}_N\big|_{T=\lambda} \frac{\exp[-s^2(T-\lambda)]}{[J_0^2(s)+Y_0^2(s)]\,s}\, ds\, d\lambda \tag{10}$$

and the initial conditions

$$z_N(0) = 1, \quad \dot{z}(0) = 0. \tag{11}$$

The solution of (10) with (11) is obtained by the following iterative method. First the integral in (10) is neglected and the resulting differential equation with (11) is integrated by use of the fourth-order Runge–Kutta method. The $\ddot{z}_N(\lambda)$ thus obtained is then substituted into the right-hand side of (10) for evaluation of the integral. The resulting inhomogeneous differential equation is then solved by the Runge–Kutta method. The same procedure is repeated until both sides of (11) evaluated by successive iterations agree to within a given small error E. This given E is then reduced by one order of magnitude and the whole computation repeated until values of $z_N(T)$ obtained for the two successive values of E agree with each other up to a given number of decimal places. The convergence of the iterative solution of a class of integro-differential equations is discussed by Phillips (1970).

For the actual numerical computation, we considered the vibration in glycerine of a steel wire of radius 0·1 cm and length 10 cm stretched between two fixed points at $S = 9{\cdot}8 \times 10^6$ dynes. The initial displacement was given by (11) with $h_N a = 0{\cdot}01$ cm. Thus $\rho_s = 7{\cdot}9\,\mathrm{g/cm^3}$, $\rho = 1{\cdot}25\,\mathrm{g/cm^3}$, $\nu = 6{\cdot}8\,\mathrm{cm^2/s}$, $l = 100$, $\Omega = 583{\cdot}9/\mathrm{s}$ and $\epsilon = 0{\cdot}2579$. Numerical results were obtained only for the initial conditions corresponding to $N = 1$ and 3. The values of $z_N(T)$ at various key points for the case $N = 3$ are given up to $T = 3{\cdot}206$ in tables 1 and 2. All computations were carried out using an IBM 360. The time interval used in the Runge–Kutta method was 10^{-3}. The trapezoidal rule was used to evaluate the integral in (10). The values of $z_N(T)$ for $N = 3$ obtained with $E = 10^{-4}$ and 10^{-6} agree up to the third decimal place. The computer time required for the case of $N = 1$ was slightly over 1 h but almost 2 h was needed for the case of $N = 3$. For higher Fourier components, the time step size required for numerical convergence decreases as N^2 for a given accuracy and the computer time required increases rapidly. Moreover, the maximum attainable T decreases because of the increase in the storage space required, for a given T, in the IBM 360, whose storage space is limited. In principle, however, the solution for given initial conditions can be obtained by superposition of sufficiently large numbers of Fourier components. For the purpose of assessing the significance of the transient effect, it suffices to consider only one Fourier component. The transient effect is assessed by comparing the present results with the results from potential theory and the theory based on the quasi-steady drag.

If the fluid is frictionless, then $\epsilon = 0$ and the right-hand side of (8) vanishes. The corresponding solution of (8) with the condition (9) is

$$Z(T, X) = h_N \cos(N\pi T)\sin(N\pi X) = h_N Y_N(T)\sin(N\pi X).$$

Thus the string oscillates with a constant frequency $N\pi$ without damping. The normalized displacements $Y_N(T)$ obtained from the above equation for the same numerical example are given in the third column of table 1.

On the other hand, if the usual quasi-steady viscous drag $D_\nu = 4\pi a^2\rho\nu\Omega Z_T$ (see Batchelor 1967, p. 357) and the added-mass drag $D_a = \pi a^2\rho a\Omega^2 Z_{TT}$ for a circular cylinder are used locally, then instead of (10) we have

$$Z_{TT} - Z_{XX} + 2\beta Z_T = 0, \tag{12}$$

where
$$\beta = \frac{4\mu}{(\rho+\rho_s)\,a\Omega(2\nu/n)^{\frac{1}{2}}},$$

T	$z_3(T)$	$Y_3(T)$	$y_3(T)$
0·000	1·0000	1·0000	1·0000
0·026	0·9698	0·9701	0·9559
0·086	0·6914	0·6891	0·6571
0·126	0·3681	0·3740	0·3498
0.156	0·0888	0·1004	0·0946
0·166	−0·0063	0·0063	0·0083
0·176	−0·1007	−0·0879	−0·0767
0·206	−0·3707	−0·3622	−0·3190
0·266	−0·7762	−0·8053	−0·6893
0·316	−0·9014	−0·9867	−0·8228
0·326	−0·8997	−0·9976	−0·8726
0·336	−0·8892	−0·9997	−0·8250
0·386	−0·7124	−0·8793	−0·7077
0·446	−0·2940	−0·4872	−0·3828
0·476	−0·0451	−0·2243	−0·1769
0·486	0·0383	−0·1316	−0·1060
0·496	0·1204	−0·0377	−0·0349
0·506	0·2006	0·0565	0·0358
0·546	0·4859	0·4209	0·3011
0·626	0·7678	0·9275	0·6454
0·636	0·7693	0·9585	0·6640
0·666	0·7276	1·0000	0·6831

TABLE 1. Displacements of a vibrating string

T	$z_3(T)$	$y_3(T)$	T	$z_3(T)$	$y_3(T)$
0·786	0·0531	0·2831	1·896	0·4409	0·1789
0·796	−0·0193	0·2268	1·996	0·2389	0·3185
0·826	−0·2288	0·0520	2·046	0·0406	0·2860
0·836	−0·2939	−0·0066	2·056	−0·0010	0·2719
0·946	−0·6759	−0·5036	2·166	−0·3523	0·0127
0·996	−0·5946	−0·5649	2·176	−0·3666	−0·0145
1·076	−0·2583	−0·4142	2·186	−0·3868	−0·0412
1·116	−0·0093	−0·2518	2·336	−0·6458	−0·2628
1·126	0·0533	−0·2509	2·366	−0·0128	−0·2493
1·166	0·2866	−0·0136	2·376	0·0232	−0·2404
1·176	0·3378	0·0345	2·496	0·3231	−0·0194
1·256	0·5824	0·3840	2·506	0·3300	0·0031
1·336	0·4582	0·4657	2·516	0·3336	0·0254
1·426	0·0202	0·2924	2·666	0·0493	0·2174
1·436	−0·0345	0·2585	2·686	−0·0136	0·2132
1·496	−0·3274	0·0270	2·836	−0·2933	0·0048
1·506	−0·3660	−0·0129	2·846	−0·2913	−0·0137
1·586	−0·5108	−0·2848	2·986	−0·0260	−0·1782
1·666	−0·3452	−0·3854	2·996	0·0014	−0·1795
1·736	−0·0361	−0·3008	3·136	0·2536	−0·0506
1·746	0·0116	−0·2776	3·166	0·2492	−0·0100
1·836	0·3633	−0·0024	3·176	0·2432	0·0053
1·856	0·4065	0·0624	3·206	0·1964	0·0495

TABLE 2. Damping of a vibrating string

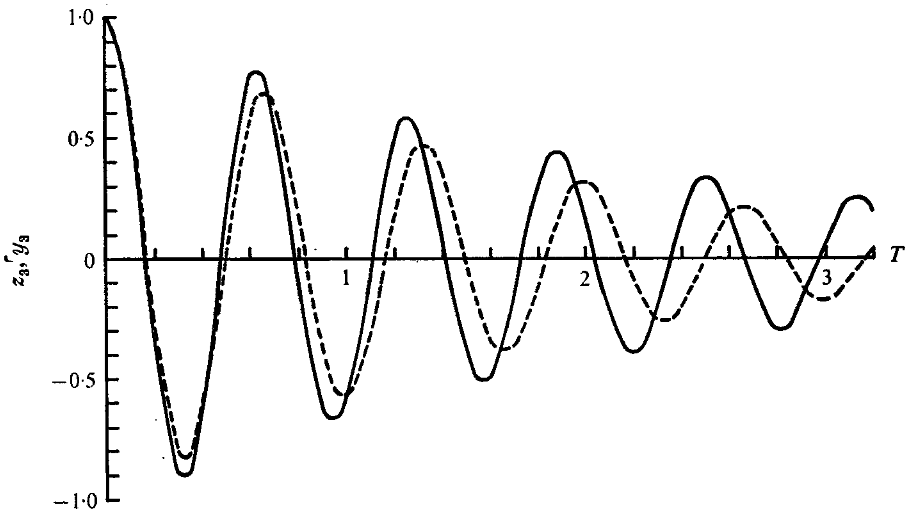

FIGURE 1. Damped vibration of a string. ———, present theory; – – – – –, quasi-steady theory.

n being the actual frequency of the oscillation. The solution of (12) with the initial condition (9) is given by

$$Z(T, X) = h_N e^{-\beta T} \cos(nT/\Omega) \sin(N\pi X) = h_N y_N(T) \sin(N\pi X), \tag{13}$$

in which

$$\frac{n}{\Omega} = [N^2\pi^2 + d^2]^{\frac{1}{2}}, \quad d = \left(\frac{\nu}{\Omega a^2}\right)\left(\frac{\rho}{\rho+\rho_s}\right).$$

Thus the quasi-steady theory gives an exponential decay of the amplitude at a constant dimensionless frequency. For the numerical example considered, $\beta = 0{\cdot}5720$ and $n/(2\pi\Omega) = 1{\cdot}4973$, which is smaller than the frequency $\frac{3}{2}$ predicted by the potential theory only by 1·8 %. The numerical values of the normalized displacements $y_3(T)$ obtained from (13) are given in tables 1 and 2 and plotted in figure 1. The corresponding displacements $z_3(T)$ predicted by the present theory are also plotted in the same figure.

It may be seen from figure 1 that the quasi-steady theory overestimates considerably both the period and the damping rate of the oscillation. The periods predicted by the present theory for the first five cycles are 0·64, 0·62, 0·64, 0·62 and 0·62, which are approximately 7 % shorter than the constant period 0·6698 predicted by the quasi-steady theory. The dimensional frequency corresponding to a dimensionless period of 0·62 is $n = 0{\cdot}62\,(2\pi/\Omega) = 5823{\cdot}64$/s. Thus the present numerical example, with $na^2/\nu = 8{\cdot}56$ and $h_N = 0{\cdot}1$, belongs to the second limiting case mentioned in the last section. The amplitude of the vibration varies during the first five cycles with consecutive amplitude ratios 0·7693, 0·7574, 0·7570, 0·7566 and 0·7584. It is reasonable to expect that the amplitude ratio will increase slightly as T increases and reach an asymptotic value. However it is very unlikely that this asymptotic value will approach the considerably smaller ratio $\exp(-0{\cdot}5720/1{\cdot}493) = 0{\cdot}6825$ predicted by the quasi-steady theory. Figure 1 should not be interpreted as showing good agreement between the quasi-steady theory and the present theory during the first quarter-cycle. In fact table 1 shows that the results predicted by potential theory agree more closely with the present results during the initial stage of vibration as expected.

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Appendix. Forced oscillation of a circular cylinder

Consider the flow around a circular cylinder which, after starting suddenly from rest, performs simple harmonic oscillations with a finite frequency ω and an amplitude h. For this case, $v(\tau) = h\sin\omega\tau\, H(\tau)$, where $H(\tau)$ is the Heaviside unit step function, and the solution for χ is (cf. Carslaw & Jaeger 1959, p. 339)

$$\chi = \frac{1}{2i}\left[\frac{2K_0(\omega' re^{\frac{1}{4}i\pi})}{K_0(\omega' e^{\frac{1}{4}i\pi})}he^{i\omega\tau} - \text{c.c.}\right]$$
$$-\frac{4h\omega}{\pi}\int_0^\infty \exp(-s^2\tau),\ \frac{J_0(sr)Y_0(s) - J_0(s)Y_0(sr)}{(\omega^2+s^4)\,[J_0^2(s)+Y_0^2(s)]}\,s\,ds,$$

where c.c. stands for 'complex conjugate', $\omega' = \omega^{\frac{1}{2}}$ and K_0 is a modified Bessel function of zeroth order. Thus the transient flow considered is completely described by (4) with χ given by the above expression. As $\tau \to \infty$, the integral appearing in the above equation vanishes and we have from (4)

$$\psi = h\sin\theta\left[e^{i\omega\tau}\left\{\left(\frac{1}{r}-r\right)+\frac{2}{r}\int_1^r s\frac{K_0(Ms)}{K_0(M)}\,ds\right\}-\text{c.c.}\right]\Big/(2i),$$

where $M = \omega'\exp(\frac{1}{4}i\pi)$. By use of the relations (Abramowitz & Segun 1968, equations (*11.3.27*) and (9.6.26))

$$\int_1^r sK_0(Ms)\,ds = \frac{1}{M}K_1(M) - \frac{r}{M}K_1(Mr)$$

and
$$K_2(M) - K_0(M) = 2M^{-1}K_1(M),$$

we can reduce the above solution for ψ to

$$\psi = \frac{1}{2i}h\sin\theta\left[e^{i\omega\tau}\left\{-r+\frac{K_2(M)}{K_0(M)}\frac{1}{r}-\frac{2}{MK_0(M)}K_1(Mr)\right\}-\text{c.c.}\right],$$

which is the known stream function for quasi-steady oscillatory flow around a cylinder (Stokes 1851; see Stuart 1963, p. 391; Holtsmark *et al.* 1954). Moreover, it can be shown that $f \sim r^{-\frac{3}{2}}$ and $g \sim r^{-\frac{1}{2}}$ as $r \to \infty$ for any $\tau > 0$.

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